

# Melvin solution in string theory

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## Abstract

We identify string theory counterpart of the dilatonic Melvin  $D = 4$  background describing a “magnetic flux tube” in low-energy field theory limit. The corresponding  $D = 5$  bosonic string model containing extra compact Kaluza-Klein dimension is a direct product of the  $D = 2$  Minkowski space and a  $D = 3$  conformal  $\sigma$ -model. The latter is a singular limit of the  $[SL(2, R) \times R]/R$  gauged WZW theory. This implies, in particular, that the dilatonic Melvin background is an exact string solution to all orders in  $\alpha'$ . Moreover, the  $D = 3$  model is formally related by an abelian duality to a flat space of non-trivial topology. The conformal field theory for the Melvin solution is exactly solvable (and for special values of magnetic field parameter is equivalent to CFT for a  $Z_N$  orbifold of 2-plane times a circle) and should exhibit tachyonic instabilities.

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1. There are two simple analogues of a uniform Maxwell magnetic field background in the Einstein-Maxwell theory: one is the Robinson-Bertotti solution [1], i.e.  $(\text{AdS})_2 \times S^2$  with covariantly constant magnetic field  $F = B_0 \sin \theta d\theta \wedge d\varphi$  (“monopole”) on  $S^2$ , and another is the static cylindrically symmetric Melvin magnetic universe (or “flux tube”) [2]. The latter solution has  $R^4$  topology and may be used to describe a background magnetic field. Several interesting features of the Melvin solution in the context of Kaluza-Klein (super)gravity (e.g. instability against monopole or magnetic black hole pair creation) were emphasized in [3] (see also [4]).

Both solutions have direct generalisations to low-energy string theory (heterotic string or  $D = 5$  bosonic string compactified to  $D = 4$ ). In addition to the metric and the vector field the string effective action includes also the dilaton  $\phi$  and the antisymmetric tensor  $B_{\mu\nu}$ . The analogue of the Melvin solution [5] has  $B_{\mu\nu} = 0$  but a non-constant dilaton. As was noted recently [6,7], in the context of string theory there is also *another* natural uniform magnetic field solution. For  $D = 4$  its metric  $(ds^2 = -(dt + A_i dx^i)^2 + dx_i dx^i + dz^2, \quad A_i = -\frac{1}{2} F_{ij} x^j, \quad i = 1, 2)$  is that of a product of a real line  $R$  and the Heisenberg group space  $H_3$ , the dilaton is constant but the antisymmetric tensor field strength is non-trivial and is equal to the (covariantly) constant magnetic field  $H_{0ij} = F_{ij} = \text{const}$ . This background is an exact string solution [6] and the corresponding conformal string model is easily solvable [7]. The Robinson-Bertotti solution also has an exact string counterpart [8] which is a product of the two conformal theories: “ $(\text{AdS})_2$ ” ( $SL(2, R)/Z$  WZW) and “monopole” ( $SU(2)/Z_m$  WZW) [9] ones.<sup>1</sup>

This raises the question we address below: which is an exact string counterpart of the leading-order dilatonic Melvin background? Starting with the  $D = 5$  bosonic string effective action and assuming that one spatial dimension  $y = x^5$  is compactified on a circle, one finds the following dimensionally reduced  $D = 4$  action

$$S_4 = \int d^4x \sqrt{\hat{G}} e^{-2\phi+\sigma} \left[ \hat{R} + 4(\partial_\mu \phi)^2 - 4\partial_\mu \phi \partial^\mu \sigma \right. \\ \left. - \frac{1}{12} (\hat{H}_{\mu\nu\lambda})^2 - \frac{1}{4} e^{2\sigma} (F_{\mu\nu}(\mathcal{A}))^2 - \frac{1}{4} e^{-2\sigma} (F_{\mu\nu}(\mathcal{B}))^2 + O(\alpha') \right], \quad (1)$$

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<sup>1</sup> Kaluza-Klein monopoles in string theory were considered in [10]. To use  $SU(1,1)$  WZW model to construct (electro)magnetic string backgrounds by dimensional reduction was suggested in [11]. An exact  $D = 3$  monopole-type [12] magnetic background based on  $SU(2)$  WZW model tensored with linear dilaton was discussed in [13].

where

$$\hat{G}_{\mu\nu} \equiv G_{\mu\nu} - G_{55}\mathcal{A}_\mu\mathcal{A}_\nu, \quad G_{55} \equiv e^{2\sigma}, \quad \hat{H}_{\lambda\mu\nu} = 3\partial_{[\lambda}B_{\mu\nu]} - 3\mathcal{A}_{[\lambda}F_{\mu\nu]}(\mathcal{B}), \quad (2)$$

$$F_{\mu\nu}(\mathcal{A}) = 2\partial_{[\mu}\mathcal{A}_{\nu]}, \quad F_{\mu\nu}(\mathcal{B}) = 2\partial_{[\mu}\mathcal{B}_{\nu]}, \quad \mathcal{A}_\mu \equiv G^{55}G_{\mu 5}, \quad \mathcal{B}_\mu \equiv B_{\mu 5}.$$

For both Melvin solution [5] and the solution of [6]  $\sigma = 0$  so that the two vector fields  $\mathcal{A}_\nu$  and  $\mathcal{B}_\nu$  are equal (up to sign). Then (1) reduces to

$$S_4 = \int d^4x \sqrt{\hat{G}} e^{-2\phi} \left[ \hat{R} + 4(\partial_\mu\phi)^2 - \frac{1}{12}(H_{\mu\nu\lambda})^2 - \frac{1}{2}(F_{\mu\nu}(A))^2 + O(\alpha') \right], \quad (3)$$

where we have set  $A_\mu = \mathcal{A}_\mu = -\mathcal{B}_\mu$  (the Chern-Simons term vanishes in the axially symmetric case we consider so we omit the hat on  $H$ ). In this string frame the dilatonic Melvin solution has a simple form [5] (here  $D = 4$ ,  $a = 1$ )<sup>2</sup>

$$ds_4^2 = \hat{G}_{\mu\nu}dx^\mu dx^\nu = -dt^2 + dz^2 + d\rho^2 + \Lambda^{-2}(\rho)\rho^2 d\varphi^2, \quad (4)$$

$$A = A_\mu dx^\mu = b\Lambda^{-1}(\rho)\rho^2 d\varphi, \quad F_{\mu\nu}^2 = 8b^2\Lambda^{-2}(\rho), \quad (5)$$

$$\phi = \phi_0 - \frac{1}{2}\ln\Lambda(\rho), \quad \Lambda(\rho) \equiv 1 + b^2\rho^2, \quad b = \text{const}. \quad (6)$$

The parameter  $b$  determines the strength of the magnetic field. The curvature, magnetic field and the effective string coupling  $e^\phi$  have maxima at  $\rho = 0$  and decrease to zero at infinity.<sup>3</sup> From the point of view of the low-energy field theory one could formally use also the gauge potential

$$A' = A - b^{-1}d\varphi = -b^{-1}\Lambda^{-1}(\rho)d\varphi, \quad (7)$$

which has the same field strength but is singular on the  $\rho = 0$  axis (the gauge transformation is singular at  $\rho = 0$ ). The string models corresponding to  $A$  and  $A'$ , in general, will *not* be equivalent (see below).

For comparison, the constant magnetic field solution of (3) found in [6] has the following explicit form

$$ds_4^2 = -(dt + b\rho^2 d\varphi)^2 + dz^2 + d\rho^2 + \rho^2 d\varphi^2, \quad (8)$$

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<sup>2</sup> In the Einstein frame the magnetic pressure of the flux tube is balanced by the gravitational and dilatonic attraction. In the string frame the dilatonic contribution has the same sign as the magnetic field one.

<sup>3</sup> Since the radius of the circular  $\varphi$  dimension approaches zero at infinity this solution was also interpreted in [5] as a “Minkowski membrane” with the  $(\rho, \varphi)$ -space (a “bottle” with an infinite narrowing neck) playing the role of a non-compact internal 2-space.

$$A = b\rho^2 d\varphi , \quad F_{\mu\nu}^2 = 8b^2 , \quad (9)$$

$$B = \frac{1}{2}B_{\mu\nu}dx^\mu \wedge dx^\nu = b\rho^2 d\varphi \wedge dt , \quad H = F \wedge dt , \quad \phi = \text{const} . \quad (10)$$

While the metric (4) is static and is a product of the  $D = 2$  Minkowski space and a curved  $D = 2$  part, the metric (8) is stationary with a non-trivial part being 3-dimensional.

2. As in the case of (8),(9),(10) discussed in [6,7] we can now write down the  $D = 5$  string  $\sigma$ -model which corresponds to the dilatonic Melvin solution (4) (considered as a  $D = 4$  “image” of a  $D = 5$  bosonic string background). Using the correspondence between (1) and (3) we get

$$I_5 = \frac{1}{\pi\alpha'} \int d^2\sigma [(\hat{G}_{\mu\nu} + B_{\mu\nu})\partial x^\mu \bar{\partial} x^\nu + e^{2\sigma}[\partial y + \mathcal{A}_\mu(x)\partial x^\mu][\bar{\partial} y + \mathcal{A}_\nu(x)\bar{\partial} x^\nu] \quad (11)$$

$$+ \mathcal{B}_\mu(x)(\partial x^\mu \bar{\partial} y - \bar{\partial} x^\mu \partial y) + \mathcal{R}\phi(x)] , \quad \mathcal{R} \equiv \frac{1}{4}\alpha' R^{(2)} ,$$

i.e.,

$$I_5 = \frac{1}{\pi\alpha'} \int d^2\sigma ( - \partial t \bar{\partial} t + \partial z \bar{\partial} z + \partial \rho \bar{\partial} \rho + \Lambda^{-2}(\rho) \rho^2 \partial \varphi \bar{\partial} \varphi \quad (12)$$

$$+ [\partial y + b\Lambda^{-1}(\rho) \rho^2 \partial \varphi][\bar{\partial} y + b\Lambda^{-1}(\rho) \rho^2 \bar{\partial} \varphi]$$

$$+ b\Lambda^{-1}(\rho) \rho^2 (\bar{\partial} \varphi \partial y - \partial \varphi \bar{\partial} y) + \mathcal{R}[\phi_0 - \frac{1}{2} \ln \Lambda(\rho)] ) .$$

Here  $\varphi$  has the period  $2\pi$  and the “Kaluza-Klein” coordinate  $y$  – the period  $2\pi R$ . The action (12) can be represented as the sum of  $(t, z)$  Minkowski part and a non-trivial 3-dimensional  $(\rho, \varphi, y)$  theory

$$I_5 = \frac{1}{\pi\alpha'} \int d^2\sigma ( - \partial t \bar{\partial} t + \partial z \bar{\partial} z + \partial \rho \bar{\partial} \rho + \Lambda^{-1}(\rho) \rho^2 \partial \varphi \bar{\partial} \varphi + \partial y \bar{\partial} y \quad (13)$$

$$+ 2b\Lambda^{-1}(\rho) \rho^2 \bar{\partial} \varphi \partial y + \mathcal{R}[\phi_0 - \frac{1}{2} \ln \Lambda(\rho)] ) \equiv \frac{1}{\pi\alpha'} \int d^2\sigma ( - \partial t \bar{\partial} t + \partial z \bar{\partial} z ) + I_3 .$$

If one uses  $A'$  in (7) the action found from (11) is

$$I'_5 = \frac{1}{\pi\alpha'} \int d^2\sigma ( - \partial t \bar{\partial} t + \partial z \bar{\partial} z + \partial \rho \bar{\partial} \rho + b^{-2} \Lambda^{-1}(\rho) \partial \varphi \bar{\partial} \varphi + \partial y \bar{\partial} y \quad (14)$$

$$- 2b^{-1} \Lambda^{-1}(\rho) \bar{\partial} \varphi \partial y + \mathcal{R}[\phi_0 - \frac{1}{2} \ln \Lambda(\rho)] ) .$$

The actions (13) and (14) are formally related by the “gauge transformation”

$$y \rightarrow y' = y - b^{-1} \varphi , \quad (15)$$

which is well-defined only if the periods of  $y$  and  $b^{-1}\varphi$  are the same. Since (13) is non-singular this suggests that (14) defines a regular string model only if  $R = mR'$ ,  $b^{-1} = nR'$  ( $m, n$  are integers and  $2\pi R'$  is a period of  $y'$ ), i.e. when  $Rb = m/n$  (see also below). The string models (13) and (14) are thus equivalent ( $R = R'$ ) only if  $Rb = 1/n$ . It is the non-singular model (13) that is the string analogue of the Melvin solution.

Eq. (13) may be compared with the model

$$\begin{aligned} \bar{I}_5 = \frac{1}{\pi\alpha'} \int d^2\sigma [ & -\partial t \bar{\partial} t + \partial z \bar{\partial} z + \partial \rho \bar{\partial} \rho + \rho^2 \partial \varphi \bar{\partial} \varphi \\ & + \partial y \bar{\partial} y + 2b\rho^2 \bar{\partial} \varphi (\partial y - \partial t) + \mathcal{R}\phi_0 ] , \end{aligned} \quad (16)$$

representing the solution (8),(9),(10) [6,7]. Near  $\rho = 0$  (where the magnetic field is approximately constant) (13) reduces to (16) up to the “rotation” term  $-2b\rho^2 \bar{\partial} \varphi \partial t$ .

3. Let us now show that the non-trivial  $(\rho, \varphi, y)$  part  $I_3$  of (13) is related by a special limit to the  $[SL(2, R) \times R]/R$  gauged WZW (or “charged black string”) model [14]. A (locally) equivalent  $D = 3$  model appeared in [15] (it was obtained by gauging a  $U(1)$  subgroup of the non-semisimple  $E_2^c$  WZW model of [16]) where its formal relation to  $[SL(2, R) \times R]/R$  gauged WZW model was already observed.<sup>4</sup> In the Euler angle parametrisation of  $SL(2, R)$ ,  $g = e^{\frac{i}{2}\theta_L \sigma_2} e^{\frac{1}{2}\tilde{r}\sigma_1} e^{\frac{i}{2}\theta_R \sigma_2}$ ,  $\theta_L = \theta + \tilde{\theta}$ ,  $\theta_R = \tilde{\theta} - \theta$ , the action of the  $[SL(2, R) \times R]/R$  gauged WZW model is given by (see e.g. [17])

$$\begin{aligned} I_{g_{wzw}}(r, \theta, \tilde{\theta}) = \frac{k}{\pi} \int d^2\sigma ( & \frac{1}{4} \partial r \bar{\partial} r + (1+q)[1 - 2(1+q)X^{-1}(r)] \partial \theta \bar{\partial} \theta \\ & - q[1 - 2qX^{-1}(r)] \partial \tilde{\theta} \bar{\partial} \tilde{\theta} - 2q(q+1)X^{-1}(r) (\partial \theta \bar{\partial} \tilde{\theta} - \partial \tilde{\theta} \bar{\partial} \theta) + \mathcal{R}[\phi'_0 - \frac{1}{2} \ln X(r)]), \end{aligned} \quad (17)$$

$$X(r) \equiv \cosh r + 1 + 2q .$$

The free parameter  $q$  determines the embedding of  $R$  into  $SL(2, R) \times R$ . It is easy to see that  $I_3$  in (13) can be formally obtained from (17) in the following special limit (in this limit  $X(r) \rightarrow 2b^{-2}\epsilon^2\Lambda(\rho)$ )<sup>5</sup>

$$k = (\alpha'\epsilon^2)^{-1} \rightarrow \infty , \quad q = -1 + b^{-2}\epsilon^2 \rightarrow -1 , \quad e^{\phi'_0} = 2b^{-2}\epsilon^2 e^{\phi_0} , \quad \epsilon \rightarrow 0 , \quad (18)$$

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<sup>4</sup> I am grateful to K. Sfetsos for pointing this out to me while the present paper was in preparation.

<sup>5</sup> The mass and axionic charge of the black string [14] are related to  $k$  and  $q$  by  $M = q/\sqrt{k}$ ,  $Q^2 = q(1+q)/k$  and thus vanish in this limit but the simultaneous singular rescalings of the coordinates give rise to a non-trivial model (see also [15]).

$$r = 2\epsilon\rho, \quad \tilde{\theta} = \epsilon[-(q+1)/q]^{1/2}y \rightarrow b^{-1}\epsilon^2y, \quad (19)$$

$$\theta = \varphi + \frac{q}{q+1}\tilde{\theta} = \varphi + \epsilon[-q/(q+1)]^{1/2} \rightarrow \varphi + by.$$

This limit does *not*, however, respect the *global* structure of the two models: since  $\theta, \tilde{\theta}$  and  $\varphi$  should have periods  $2\pi$ , the radius  $R$  of  $y$  should satisfy  $b^{-1}\epsilon^2R = m$  and  $bR = n$  which is not possible for integer  $m$  and  $n$ . Still, it preserves the local operator relations of the coset CFT (like the expression for the stress tensor in terms of currents and OPE's), i.e. gives rise to analogous relations for the theory (13). The CFT corresponding to (13) is, in fact, much simpler than (any regular limit of)  $[SL(2, R) \times R]/R$  coset model. For example, its central charge has the free-theory value (since  $c_{g_{WZW}} = 3k/(k-2)$  does not depend on  $q$  its limit is simply the  $k \rightarrow \infty$  one). The existence of this formal limiting relation is also sufficient to argue that (13) is a conformal  $\sigma$ -model. The fact that in the limit (18)  $k \rightarrow \infty$  does not by itself imply that all higher loop corrections to  $\beta$ -functions should automatically vanish since they may depend on  $q$  in a general scheme (see [17]). However, for any  $q$  there exists a scheme in which the leading-order background corresponding to (17) is exact to all orders.<sup>6</sup> We thus conclude that (in such scheme) the dilatonic Melvin solution (4),(5),(6) (embedded in bosonic string theory according to (12)) is an *exact string solution*.

3. A simple structure of the Melvin model (suggested by its  $k \rightarrow \infty, q \rightarrow -1$  relation to the  $[SL(2, R) \times R]/R$  coset model) is further revealed by making abelian duality transformations [22]. This helps to construct explicitly the corresponding conformal field theory since abelian duality in the direction of a compact isometry gives a  $\sigma$ -model representing an equivalent CFT [23,24]. We shall see that duality in the angular coordinate of the plane leads to non-trivial models when combined with shifts in the compact Kaluza-Klein dimension.

The action (13) has two non-trivial symmetries under constant shifts of  $y$  and  $\varphi$ . Like (16) the model (13) is “self-dual” with respect to duality in the  $y$ -direction (the dual action has the same form with  $y$  replaced by the dual coordinate  $\tilde{y}$  with period  $2\pi\tilde{R}$ ,  $\tilde{R} = \alpha'/R$ ).

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<sup>6</sup> For the  $[SL(2, R) \times R]/R$  model this was explicitly checked at the two-loop order in [17] and was also suggested in the context of conformal perturbation theory in [18]. Similar statement holds for other gauged WZW backgrounds as was demonstrated to all orders on the example of  $SL(2, R)/R$  in [19] and argued to be true in general in [20,21].

The duality transformation in the  $\varphi$  direction can be performed by gauging the symmetry  $\varphi \rightarrow \varphi + a$  of (13) [23,24]

$$I_3 = \frac{1}{\pi\alpha'} \int d^2\sigma (\partial\rho\bar{\partial}\rho + \Lambda^{-1}(\rho)\rho^2(\partial\varphi + V)(\bar{\partial}\varphi + \bar{V}) + \partial y\bar{\partial}y) \quad (20)$$

$$+ 2b\Lambda^{-1}(\rho)\rho^2(\bar{\partial}\varphi + \bar{V})\partial y + \alpha'(\bar{V}\partial\tilde{\varphi} - V\bar{\partial}\tilde{\varphi}) + \mathcal{R}[\phi_0 - \frac{1}{2}\ln\Lambda(\rho)] ,$$

where  $(V, \bar{V})$  is a 2d gauge field and  $\tilde{\varphi}$  is a Lagrange multiplier. Fixing the gauge  $\varphi = 0$  and integrating out  $V, \bar{V}$  we find the dual model (see (6))

$$\tilde{I}_3 = \frac{1}{\pi\alpha'} \int d^2\sigma [\partial\rho\bar{\partial}\rho + \alpha'^2(\rho^{-2} + b^2)\partial\tilde{\varphi}\bar{\partial}\tilde{\varphi} + \partial y\bar{\partial}y + 2\alpha'b\partial y\bar{\partial}\tilde{\varphi} + \mathcal{R}(\phi_0 - \ln\rho)] \quad (21)$$

$$= \frac{1}{\pi\alpha'} \int d^2\sigma [\partial\rho\bar{\partial}\rho + \alpha'^2\rho^{-2}\partial\tilde{\varphi}\bar{\partial}\tilde{\varphi} + \partial\bar{y}\bar{\partial}\bar{y} + \alpha'b(\partial\bar{y}\bar{\partial}\tilde{\varphi} - \bar{\partial}\bar{y}\partial\tilde{\varphi}) + \mathcal{R}(\tilde{\phi}_0 - \ln\rho)] , \quad (22)$$

$$\bar{y} \equiv y + \alpha'b\tilde{\varphi} .$$

To be able to argue [23] that  $I_3$  in (13) and  $\tilde{I}_3$  in (21) represent equivalent conformal theories we need to impose the condition that  $\tilde{\varphi}$  has the same period as  $\varphi$ , i.e.  $2\pi$ . Then  $\bar{y}$  is a well-defined periodic coordinate only if  $\alpha'b/R$  is rational. In this case we may assume that  $\bar{y}$  is inert under further duality rotation of  $\tilde{\varphi}$  into  $\varphi'$  so that we end up with a flat space model which is a direct product of a 2-plane and a circle<sup>7</sup>

$$\tilde{I}_3 = \frac{1}{\pi\alpha'} \int d^2\sigma [\partial\rho\bar{\partial}\rho + \rho^2\partial\bar{\varphi}\bar{\partial}\bar{\varphi} + \partial\bar{y}\bar{\partial}\bar{y} + \mathcal{R}\tilde{\phi}_0] , \quad \bar{\varphi} \equiv \varphi' + by . \quad (23)$$

A shift of  $\varphi'$  is due to a constant antisymmetric tensor term in (22). The condition that  $\bar{\varphi}$  has period  $2\pi$  is  $bR = \text{integer}$ . This is consistent with  $\alpha'b/R = \text{rational}$  only if  $\alpha'b^2$  is also rational.

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<sup>7</sup> A similar conclusion is reached for the model (16) by making a duality rotation  $\varphi \rightarrow \tilde{\varphi}$ , a shift of  $y$  and  $t$  by  $\alpha'b\tilde{\varphi}$  and duality rotation of  $\tilde{\varphi}$  “backwards” (see [25,26] where  $y$  was non-compact). Analogous observation for the model (13) with a non-compact  $y$  was made also in [15], see footnote 4. Note in this connection that the  $[SL(2, R) \times R]/R$  WZW model can be obtained by an  $O(2, 2)$  duality rotation (with  $q$  as a parameter) from the ungauged  $SL(2, R)$  WZW model [23,27]. It may be useful also to recall that the model of [6,7] is closely related to a non-semisimple  $E_2^c$  WZW model of [16] which itself is a formal singular limit [15,28] of the  $SU(2) \times R$  WZW model. This suggests that the models (13) and (16) are special cases of a more general class of string solutions (this indeed turns out to be the case and will be discussed in [29]).

If instead of (13) one starts with the “singular gauge” model (14) and makes the duality rotation of  $\varphi$  one directly obtains a flat model with constant antisymmetric tensor and dilaton

$$\tilde{I}'_3 = \frac{1}{\pi\alpha'} \int d^2\sigma [\partial\rho\bar{\partial}\rho + \alpha'^2 b^2 (1 + b^2 \rho^2) \partial\tilde{\varphi}\bar{\partial}\tilde{\varphi} + \partial y\bar{\partial}y - 2\alpha' b \partial y\bar{\partial}\tilde{\varphi} + \mathcal{R}(\tilde{\phi}_0 + \ln b)] \quad (24)$$

$$= \frac{1}{\pi\alpha'} \int d^2\sigma [\partial\rho\bar{\partial}\rho + \alpha'^2 b^4 \rho^2 \partial\tilde{\varphi}\bar{\partial}\tilde{\varphi} + \partial\hat{y}\bar{\partial}\hat{y} - \alpha' b (\partial\hat{y}\bar{\partial}\tilde{\varphi} - \bar{\partial}\hat{y}\partial\tilde{\varphi}) + \mathcal{R}\tilde{\phi}_0] , \quad (25)$$

$$\hat{y} \equiv y - \alpha' b \tilde{\varphi} , \quad \tilde{\phi}_0 \equiv \phi_0 + \ln b .$$

This model is equivalent to (14) if  $\tilde{\varphi}$  has period  $2\pi$ . Then  $\hat{y}$  is a globally defined compact coordinate provided  $\alpha' b/R$  =rational. At the same time, (13) and (14) are equivalent if  $Rb = 1/n$ , i.e. if  $\alpha' b^2$  is rational, in agreement with what we have found from (23). In particular, we conclude that in the special case of  $R = \alpha' b m/n$ ,  $\alpha' b^2 = 1/N$  ( $m, n, N$ =integers) the Melvin CFT (13) is equivalent to that of an orbifold of 2-plane  $R^2/Z_N$  times a circle. The  $R^2/Z_N$  model is equivalent to the “string on the cone” (or “string on cosmic string background”, or “string in Rindler space at finite temperature”) model recently discussed and explicitly solved in [30,31] (see also [32,33]). The magnetic field strength parameter  $b$  is related to the Rindler “temperature” by  $\beta = 2\pi\alpha' b^2 = 2\pi/N$ ,  $N = 1, 2, \dots$ . It was found in [30,31] that this model contains tachyons (both in the bosonic and in the superstring versions) with masses  $\alpha' M^2 = -4 + 4/N$ . The same conclusion is thus true for the Melvin model with special value of the magnetic field strength parameter  $\alpha' b^2 = 1/N$ : it is *unstable* when considered as a string theory solution.<sup>8</sup> Similar tachyonic instability was found also for the constant magnetic field solution (8),(9),(10) in [7]. This suggests that uniform magnetic field backgrounds are in general unstable in string theory.

Let us now discuss the case of generic parameters  $b$  and  $R$  starting with the model (14) or, equivalently, its dual (24). The 3-metric in (24) is

$$ds^2 = d\rho^2 + \alpha'^2 b^4 \rho^2 d\tilde{\varphi}^2 + d(y - \alpha' b \tilde{\varphi})^2 = d\rho^2 + R^2 |d\psi + \tau(\rho)\tilde{\varphi}|^2 , \quad (26)$$

$$y \equiv R\psi , \quad \tau \equiv -(1 + ib\rho)\alpha' b/R .$$

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<sup>8</sup> Instability of the Melvin solution (and related absence of residual supersymmetry in the context of supergravity) was discussed in the field-theory limit in [3].



Its constant  $\rho$  section is a 2-torus ( $\psi$  and  $\tilde{\varphi}$  have periods  $2\pi$ ) with  $\rho$ -dependent modulus. This metric is regular and *flat* for  $\rho \neq 0$  but degenerates at  $\rho = 0$  where the torus develops a pinch ( $\text{Im } \tau \rightarrow 0$ ). The classical string equations which follow from (24) are

$$\partial\bar{\partial}\rho - \alpha'^2 b^4 \rho \partial\tilde{\varphi}\bar{\partial}\tilde{\varphi} = 0, \quad \partial(\rho^2\bar{\partial}\tilde{\varphi}) + \bar{\partial}(\rho^2\partial\tilde{\varphi}) = 0, \quad \partial\bar{\partial}(y - \alpha'b\tilde{\varphi}) = 0. \quad (27)$$

The  $\rho, \tilde{\varphi}$  equations are thus the same as in the case of the cone theory  $L = \partial\rho\bar{\partial}\rho + \alpha'^2 b^4 \rho^2 \partial\tilde{\varphi}\bar{\partial}\tilde{\varphi}$ . This suggests that this theory is not well-defined for generic  $b$ .

The 3-metric corresponding to the model (21) dual to (13) is

$$ds^2 = d\rho^2 + \alpha'\rho^{-2}d\tilde{\varphi}^2 + d(y + \alpha'b\tilde{\varphi})^2 = d\rho^2 + R^2|d\psi + \tau(\rho)\tilde{\varphi}|^2, \quad (28)$$

$$y \equiv R\psi, \quad \tau \equiv (b + i\rho^{-1})\alpha'/R.$$

This is a curved space which is an obvious generalisation of a 2-space dual to 2-plane. Indeed, the classical string equations for this model

$$\partial\bar{\partial}\rho + \alpha'^2 \rho^{-3} \partial\tilde{\varphi}\bar{\partial}\tilde{\varphi} = 0, \quad \partial(\rho^{-2}\bar{\partial}\tilde{\varphi}) + \bar{\partial}(\rho^{-2}\partial\tilde{\varphi}) = 0, \quad \partial\bar{\partial}(y + \alpha'b\tilde{\varphi}) = 0, \quad (29)$$

are essentially the same (for  $\rho$  and  $\tilde{\varphi}$ ) as for the model  $\tilde{L} = \partial\rho\bar{\partial}\rho + \alpha'^2 \rho^{-2} \partial\tilde{\varphi}\bar{\partial}\tilde{\varphi}$  and thus can be solved explicitly in terms of free fields by observing that duality maps classical string solutions into solutions of a dual model ( $G_{\mu\nu}\partial_a x^\nu = \epsilon_{ab}\partial^b \tilde{x}_\mu$ , etc). Indeed, if  $\rho_0$  and  $\varphi_0$  are solutions of the flat 2-space model  $L = \partial\rho\bar{\partial}\rho + \rho^2\partial\varphi\bar{\partial}\varphi$  (expressed in terms of free fields  $x_1, x_2$  as follows  $\rho_0^2 = x_1^2 + x_2^2$ ,  $\tan\varphi_0 = x_1/x_2$ ), then the solution corresponding to  $\tilde{L}$  is  $(\rho_0, \tilde{\varphi}_0)$ , where  $\alpha'\partial_a\tilde{\varphi}_0 = \epsilon_{ab}r_0^2\partial^b\varphi_0 = \epsilon_{ab}(x_1\partial^bx_2 - x_2\partial^bx_1)$ . Then also  $\partial^a\partial_a y = -2b\epsilon_{ab}\partial^ax_1\partial^bx_2$ . Thus (29) can be solved in terms of free fields and the corresponding conformal quantum theory can be constructed explicitly like it was done in the case of (16) in [7] (this will be discussed in detail in [29]). In contrast to (14),(24) the model (13),(24) is well-defined for generic  $b$  and  $R$  (but still is unstable due to tachyons).

4. The direct superstring or heterotic string analogue of the bosonic Melvin solution is represented by (1,1) or (0,1) supersymmetric extension of the model (13) (with an internal gauge field background added in the heterotic case to ensure the global (1,1) supersymmetry, i.e. the embedding into the superstring). As in the bosonic case, the resulting models have the magnetic field being of a Kaluza-Klein origin.

One may construct a different ( $D = 4$ ) heterotic string version of the Melvin solution for which the magnetic field belongs to the internal gauge sector. The idea is to “fermionise”

the internal coordinate  $y$  in (13). Following the approach used in [7] let us consider first the  $(0, 1)$  super-extension of (13) in the  $(\rho, \varphi)$  directions (adding the corresponding left fermions) and then “fermionise”  $y$  to get the internal right Weyl fermion coupled to  $A$  in (5) (and an extra free left fermion).<sup>9</sup> The resulting  $(0, 1)$  supersymmetric heterotic  $\sigma$ -model<sup>10</sup>

$$I_{(0,1)} = \frac{1}{\pi\alpha'} \int d^2\sigma \left[ -\partial t \bar{\partial} t + \partial z \bar{\partial} z + \partial \rho \bar{\partial} \rho + \Lambda^{-2}(\rho) \rho^2 \partial \varphi \bar{\partial} \varphi - \lambda_L^t \partial \lambda_L^t + \lambda_L^z \partial \lambda_L^z \right. \\ \left. + \bar{\lambda}_L [\partial - i(\Lambda^{-1} - 2\Lambda^{-2})(\rho) \partial \varphi] \lambda_L + \bar{\psi}_R [\bar{\partial} - ie_0 b \Lambda^{-1}(\rho) \rho^2 \bar{\partial} \varphi] \psi_R \right. \\ \left. + 2e_0 b \Lambda^{-1}(\rho) \bar{\psi}_R \psi_R \bar{\lambda}_L \lambda_L \right], \quad e_0 \equiv R^{-1} = \sqrt{2/\alpha'}, \quad (30)$$

should be conformal to all orders. The fact that the duality transformation in (20) is performed only in the “left sector” (in  $\varphi$ -direction) suggests that for special  $b, R$  (30) is equivalent to the heterotic counterpart of an orbifold of a flat space.

The embedding of the Melvin solution into string theory we discussed above and the existence of a similar  $D = 5$  representation [34] for the extremal magnetic dilatonic black hole background [5,35]<sup>11</sup>

$$I = \frac{1}{\pi\alpha'} \int d^2\sigma \left[ -\partial t \bar{\partial} t + F^2(r) \partial r \bar{\partial} r + r^2 \partial \theta \bar{\partial} \theta + r^2 \sin^2 \theta \partial \varphi \bar{\partial} \varphi \right. \\ \left. + (\partial y + M \cos \theta \partial \varphi)(\bar{\partial} y + M \cos \theta \bar{\partial} \varphi) + M \cos \theta (\partial \varphi \bar{\partial} y - \bar{\partial} \varphi \partial y) + \mathcal{R}\phi(r) \right], \\ \phi = \phi_0 + \frac{1}{2} \ln F(r), \quad F(r) \equiv \left(1 + \frac{2M}{r}\right)^{-1}, \quad (31)$$

suggests that similar conformal string model exists for the dilatonic Ernst-type solution [4] which describes a pair of (extremal) magnetic black holes in a magnetic field background and thus generalises both the dilatonic Melvin and (extremal) dilatonic magnetic black hole solutions. Constructing such model may open a way to a computation of a magnetic black hole pair creation rate [4] at the level of string theory.

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<sup>9</sup> An attempt to get (as in the case of the “monopole” theory in [9]) both the left and right fermions (coupled to a spin connection of the  $(\rho, \varphi)$  metric and to the internal gauge field) from the single field  $y$  does not work in the present case ((13) does not lead to the necessary Thirring-type interaction; also, the vierbein connection for the  $(\rho, \varphi)$  part of (13)  $\omega^1_2 = (\Lambda^{-1} - 2\Lambda^{-2})d\varphi$  does not match the magnetic field  $A$  in (5)).

<sup>10</sup> Here  $\psi_R$  and  $\lambda_L = (\lambda_L^1 + i\lambda_L^2)/\sqrt{2}$  ( $\lambda_L^a$  correspond to the vierbein basis  $e^1 = d\rho$ ,  $e^2 = \rho\Lambda^{-1}(\rho)d\varphi$ ) are complex Weyl spinors. We ignore extra free fermionic terms.

<sup>11</sup> To “supersymmetrically” embed this model into the heterotic string theory one needs also to add the internal gauge field background [36].

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